# Signed Shape Tilings of Squares *

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**Abstract**

Let $T$ be a tile made up of finitely many rectangles whose corners have rational coordinates and whose sides are parallel to the coordinate axes. This paper gives necessary and sufficient conditions for a square to be tilable by finitely many $\mathbb{Q}$-weighted tiles with the same shape as $T$, and necessary and sufficient conditions for a square to be tilable by finitely many $\mathbb{Z}$-weighted tiles with the same shape as $T$. The main tool we use is a variant of F. W. Barnes's algebraic theory of brick packing, which converts tiling problems into problems in commutative algebra.

## 1 Introduction

In [3] Dehn proved that an $a \times b$ rectangle $R$ can be tiled by finitely many nonoverlapping squares if and only if $a/b$ is rational. More generally, suppose we allow the squares to have weights from $\mathbb{Z}$. An arrangement of weighted squares is a tiling of $R$ if the sum of the weights of the squares covering a region is 1 inside of $R$ and 0 outside. Dehn's argument applies in this more general setting, and shows that $R$ has a $\mathbb{Z}$-weighted tiling by squares if and only if $a/b$ is rational. In [4] this result is generalized to give necessary and sufficient conditions for a rectangle $R$ to be tilable by $\mathbb{Z}$-weighted rectangles with particular shapes. In this paper we consider a related question: Given a tile $T$ in the plane made up of finitely many weighted rectangles, is there a weighted tiling of a square by tiles with the same shape as $T$?

We define a *rectangle* in $\mathbb{R} \times \mathbb{R}$ to be a product $[b_1, b_2) \times [c_1, c_2)$ of half-open intervals, with $b_1 < b_2$ and $c_1 < c_2$. Let $A$ be a commutative ring with unity. An *$A$-weighted tile* is represented by a finite $A$-linear combination $L = a_1R_1 + \cdots + a_nR_n$ of disjoint rectangles. Associated to each such $L$ there is a function $f_L : \mathbb{R}^2 \to A$ which is supported on $\cup R_i$

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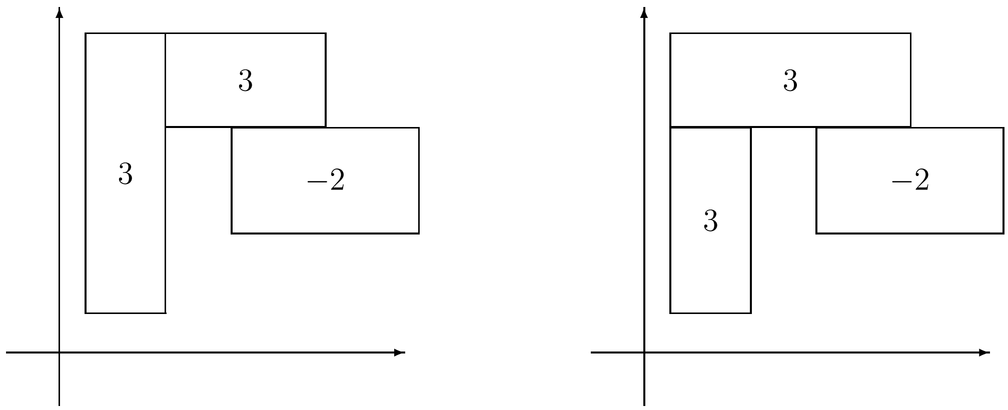


Figure 1: Two rectangle decompositions of the same $\mathbb{Z}$-weighted tile.

and whose value on $R_i$ is $a_i$. We say that $L_1$ and $L_2$ represent the same tile if $f_{L_1} = f_{L_2}$. An example of a $\mathbb{Z}$-weighted tile is given in Figure 1. We may form the sum $T_1 + T_2$ of two weighted tiles $T_1, T_2$ by superposing them in the natural way. For $a \in A$ the tile $aT$ is formed from $T$ by multiplying all the weights of $T$ by $a$. The set of all $A$-weighted tiles forms an $A$-module under these operations.

Let $U$ be an $A$-weighted tile and let $\{T_\lambda : \lambda \in \Lambda\}$ be a set of $A$-weighted tiles. We say that the set $\{T_\lambda : \lambda \in \Lambda\}$ *$A$-tiles* $U$ if there are weights $a_1, \ldots, a_n \in A$ and tiles $\tilde{T}_1, \ldots, \tilde{T}_n$, each of which is a translation of some $T_{\lambda_i}$, such that $a_1\tilde{T}_1 + \cdots + a_n\tilde{T}_n = U$. Note that we are allowed to use as many translated copies of each prototile $T_\lambda$ as we need, but we are not allowed to rotate or reflect the prototiles. Given an $A$-weighted tile $T$ and a real number $\rho > 0$ we define $T(\rho)$ to be the image of $T$ under the rescaling $(x, y) \mapsto (\rho x, \rho y)$. We say that an $A$-weighted tile $T'$ has the same shape as $T$ if there exists $\rho > 0$ such that $T'$ is a translation of $T(\rho)$. We say that $T$ *$A$-shapetiles* $U$ if $\{T(\rho) : \rho > 0\}$ $A$-tiles $U$. If $U'$ has the same shape as $U$ then $T$ $A$-shapetiles $U'$ if and only if $T$ $A$-shapetiles $U$.

In this paper we consider tiles $T$ constructed from rectangles whose corners have rational coordinates. We prove two main results about such tiles. First, we show that if $T$ is a $\mathbb{Q}$-weighted tile whose weighted area is not 0, then $T$ $\mathbb{Q}$-shapetiles a square. Second, if $T$ is a $\mathbb{Z}$-weighted tile we give necessary and sufficient conditions for $T$ to $\mathbb{Z}$-shapetile a square.

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# 2 Polynomials and tiling

Say that $T$ is a *lattice tile* if $T$ is an $A$-weighted tile made up of unit squares in $\mathbb{R}^2$ whose corners are in $\mathbb{Z}^2$. We will associate a (generalized) polynomial $f_T$ to each $A$-weighted lattice tile $T$. Our approach is similar to that used by F. W. Barnes in [2], except

that the polynomials that we construct differ from Barnes's polynomials by a factor $(X-1)(Y-1)$. Including this extra factor will allow us to generalize the construction to non-lattice tiles at the end of the section.

Our polynomials will be elements of the ring

$$A[X^{\mathbb{Z}}, Y^{\mathbb{Z}}] \;:=\; A[X, Y, X^{-1}, Y^{-1}],$$

which is naturally isomorphic to the group ring of $\mathbb{Z}\times\mathbb{Z}$ with coefficients in $A$. To begin we associate the polynomial $X^iY^j(X-1)(Y-1)$ to the unit square $S_{ij}$ with lower left corner $(i,j)\in\mathbb{Z}\times\mathbb{Z}$. Given an $A$-weighted lattice tile

$$T \;=\; \sum_{i,j} w_{ij}S_{ij},$$

by linearity we associate to $T$ the polynomial

$$f_T(X,Y) \;=\; \sum_{i,j} w_{ij}X^iY^j(X-1)(Y-1).$$

One consequence of this definition is that translating a tile by a vector $(i,j)\in\mathbb{Z}\times\mathbb{Z}$ corresponds to multiplying its polynomial by $X^iY^j$. The map $T\mapsto f_T$ gives an isomorphism between the $A$-module of $A$-weighted lattice tiles in the plane and the principal ideal in $A[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ generated by $(X-1)(Y-1)$.

**Example 2.1** Let $a,b,c,d$ be integers such that $a,b\geq 1$ and let $T$ be the $a\times b$ rectangle whose lower left corner is at $(c,d)$. Then the polynomial associated to $T$ is

$$\begin{aligned} f_T(X,Y) \;&=\; \sum_{i=c}^{c+a-1}\sum_{j=d}^{d+b-1} X^iY^j(X-1)(Y-1) \\ &=\; X^cY^d(X^a-1)(Y^b-1). \end{aligned}$$

In section 4 we will need to work with non-lattice tiles. To represent these more general tiles systematically we introduce a new set of building blocks to play the role that the unit squares $S_{ij}$ play in the theory of lattice tiles. For $\alpha,\beta\in\mathbb{R}^\times$ let $R_{\alpha\beta}$ denote the oriented rectangle with vertices $(0,0)$, $(\alpha,0)$, $(\alpha,\beta)$, $(0,\beta)$. Note that if exactly $k$ of $\alpha,\beta$ are negative then $R_{\alpha\beta}$ is equal to $(-1)^k$ times a translation of $R_{|\alpha|,|\beta|}$. We can express any rectangle in terms of the rectangles $R_{\alpha\beta}$:

**Example 2.2** Let $\alpha,\beta>0$ and let $R'_{\alpha\beta}$ be the translation of the rectangle $R_{\alpha\beta}$ by the vector $(\sigma,\tau)\in\mathbb{R}^2$. Then $R'_{\alpha\beta}=R_{\alpha+\sigma,\beta+\tau}-R_{\alpha+\sigma,\tau}-R_{\sigma,\beta+\tau}+R_{\sigma\tau}$. In particular, we have $S_{ij}=R_{i+1,j+1}-R_{i+1,j}-R_{i,j+1}+R_{ij}$.

In fact the following holds:

**Lemma 2.3** *Every $A$-weighted tile $T$ can be expressed uniquely as an $A$-linear combination of rectangles $R_{\alpha\beta}$ with $\alpha,\beta\in\mathbb{R}^\times$.*

*Proof:* By Example 2.2 every rectangle is an $A$-linear combination of the rectangles $R_{\alpha\beta}$. Therefore every $A$-weighted tile is an $A$-linear combination of the $R_{\alpha\beta}$. Suppose

$$c_1 R_{\alpha_1\beta_1} + c_2 R_{\alpha_2\beta_2} + \cdots + c_n R_{\alpha_n\beta_n} \quad = \quad 0$$

is a linear relation such that the pairs $(\alpha_i, \beta_i)$ are distinct and $c_i \neq 0$ for $1 \leq i \leq n$. Choose $j$ to maximize the distance from the origin to the far corner $(\alpha_j, \beta_j)$ of $R_{\alpha_j\beta_j}$. None of the other rectangles in the sum can overlap the region around $(\alpha_j, \beta_j)$. Since $c_j \neq 0$, this gives a contradiction. Therefore the set $\{R_{\alpha\beta} : \alpha, \beta \in \mathbb{R}^\times\}$ is linearly independent over $A$, which implies the uniqueness part of the lemma. $\square$

In order to represent arbitrary $A$-weighted tiles algebraically we introduce a generalization of the polynomials $f_T$. Let $A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ denote the set of "polynomials" with coefficients from $A$ where the exponents of $X$ and $Y$ are allowed to be arbitrary real numbers. The natural operations of addition and multiplication make $A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ a commutative ring with unity. The ring $A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ is naturally isomorphic to the group ring of $\mathbb{R} \times \mathbb{R}$ with coefficients in $A$, and contains $A[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ as a subring.

For $\alpha, \beta \in \mathbb{R}^\times$ define $f_{R_{\alpha\beta}} = (X^\alpha - 1)(Y^\beta - 1) \in A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$. By Lemma 2.3 this definition extends linearly to give a well-defined element $f_T \in A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ associated to any $A$-weighted tile $T$. It follows from Example 2.2 that this definition agrees with that given earlier if $T = S_{ij}$ is a unit lattice square, and hence also if $T$ is any lattice tile. The map $T \mapsto f_T$ gives an isomorphism between the $A$-module of $A$-weighted tiles and an $A$-submodule of $A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$. The next lemma implies that this $A$-submodule is actually an ideal in $A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$.

**Lemma 2.4** *Let $T$ be an $A$-weighted tile and let $T'$ be the translation of $T$ by the vector $(\sigma, \tau) \in \mathbb{R} \times \mathbb{R}$. Then $f_{T'} = X^\sigma Y^\tau f_T$.*

*Proof:* Let $R'_{\alpha\beta}$ be the translation of $R_{\alpha\beta}$ by $(\sigma, \tau)$. Using Example 2.2 we get

$$f_{R'_{\alpha\beta}} \quad = \quad X^\sigma Y^\tau (X^\alpha - 1)(Y^\beta - 1) \quad = \quad X^\sigma Y^\tau f_{R_{\alpha\beta}},$$

so the lemma holds for $T = R_{\alpha\beta}$. Therefore by Lemma 2.3 the lemma holds for all tiles $T$. $\square$

The next result gives a further relation between ideals and tiling.

**Proposition 2.5** *Let $U$ be a tile, let $\{T_\lambda : \lambda \in \Lambda\}$ be a collection of tiles, and let $\tilde{I} \subset A[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ be the ideal generated by the set $\{f_{T_\lambda} : \lambda \in \Lambda\}$. Then $\{T_\lambda : \lambda \in \Lambda\}$ $A$-tiles $U$ if and only if $f_U \in \tilde{I}$.*

*Proof:* We have $f_U \in \tilde{I}$ if and only if

$$f_U(X, Y) \quad = \quad \sum_{i=1}^{k} a_i X^{\sigma_i} Y^{\tau_i} f_{T_{\lambda_i}}(X, Y)$$

for some $a_i \in A$, $\sigma_i, \tau_i \in \mathbb{R}$, and $\lambda_i \in \Lambda$. Since $X^{\sigma_i} Y^{\tau_i} f_{T_{\lambda_i}}(X, Y)$ is the polynomial associated to the translation of $T_{\lambda_i}$ by the vector $(\sigma_i, \tau_i)$, we have $f_U \in \tilde{I}$ if and only if $U = a_1 \tilde{T}_1 + \ldots + a_k \tilde{T}_k$, with $\tilde{T}_i$ a translation of $T_{\lambda_i}$. Therefore $f_U \in \tilde{I}$ if and only if $\{T_\lambda : \lambda \in \Lambda\}$ $A$-tiles $U$. $\square$

**Corollary 2.6** *Let $\{T_\lambda : \lambda \in \Lambda\}$ be a collection of lattice tiles, let $I$ be the ideal in $A[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ generated by the set $\{f_{T_\lambda} : \lambda \in \Lambda\}$, and let $U$ be a lattice tile such that $f_U \in I$. Then $\{T_\lambda : \lambda \in \Lambda\}$ $A$-tiles $U$.*

The last result in this section shows what happens to $f_T$ when we replace $T$ by a rescaling.

**Lemma 2.7** *Let $T$ be an $A$-weighted tile and let $\rho$ be a positive real number. Then $f_{T(\rho)} = f_T(X^\rho, Y^\rho)$.*

*Proof:* Let $\alpha, \beta \in \mathbb{R}^\times$. Then $R_{\alpha\beta}(\rho) = R_{\rho\alpha,\rho\beta}$ and hence

$$f_{R_{\alpha\beta}(\rho)} \;=\; (X^{\rho\alpha} - 1)(Y^{\rho\beta} - 1) \;=\; f_{R_{\alpha\beta}}(X^\rho, Y^\rho).$$

Therefore the lemma holds for $T = R_{\alpha\beta}$. It follows from Lemma 2.3 that the lemma holds for all tiles $T$. □

# 3 Tiling with rational weights

This section is devoted to proving the following theorem:

**Theorem 3.1** *Let $T$ be a $\mathbb{Q}$-weighted tile made up of rectangles whose corners all have rational coordinates. Then $T$ $\mathbb{Q}$-shapetiles a square if and only if the weighted area of $T$ is not zero.*

*Proof:* It is clear that if the weighted area of $T$ is zero then $T$ cannot shapetile a square with nonzero area. Assume conversely that $T$ has nonzero weighted area. By rescaling and translation we may assume that $T$ is a lattice tile in the first quadrant. Let $T(\mathbb{N})$ denote the set $\{T(k) : k \in \mathbb{N}\}$ of positive integer rescalings of $T$. To complete the proof of Theorem 3.1 it suffices to prove that $T(\mathbb{N})$ $\mathbb{Q}$-tiles a square. First we will prove that $T(\mathbb{N})$ $\mathbb{C}$-tiles a square; from this it will follow easily that $T(\mathbb{N})$ $\mathbb{Q}$-tiles a square.

Since $T$ is a lattice tile in the first quadrant, $f_T \in \mathbb{Q}[X, Y]$ is a polynomial in the ordinary sense. We begin by interpreting the hypothesis that the weighted area of $T$ is nonzero in terms of $f_T$.

**Lemma 3.2** *There is a polynomial $f_T^* \in \mathbb{Q}[X, Y]$ such that*

$$f_T(X, Y) \;=\; (X - 1)(Y - 1) f_T^*(X, Y).$$

*Moreover, the weighted area of $T$ is equal to $f_T^*(1, 1)$, and hence $f_T^*(1, 1) \neq 0$.*

*Proof:* Since the polynomial associated to the unit square $S_{ij}$ is

$$f_{S_{ij}}(X, Y) \;=\; X^i Y^j (X - 1)(Y - 1),$$

the lemma holds for $S_{ij}$. It follows by linearity that the lemma holds for all lattice tiles in the first quadrant. $\square$

Let $I$ denote the ideal in $\mathbb{C}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ generated by $\{f_{T(k)} : k \in \mathbb{N}\}$ and let

$$g_l(X,Y) \;=\; (X^l-1)(Y^l-1)$$

be the polynomial associated to an $l \times l$ square with lower left corner $(0,0)$. To show that $T(\mathbb{N})$ $\mathbb{C}$-tiles a square it suffices by Corollary 2.6 to show that $g_l \in I$ for some positive integer $l$. In order to get information about $I$ we consider the set $V(I) \subset \mathbb{C}^\times \times \mathbb{C}^\times$ of common zeros of the elements of $I$. The set $V(I)$ is essentially the union of the lines $X = 1$ and $Y = 1$ with the "shape variety" of $T(\mathbb{N})$ as defined by Barnes [2, §3].

We wish to determine which points $(\alpha,\beta) \in \mathbb{C}^\times \times \mathbb{C}^\times$ might be in $V(I)$. Let $m$ be the $X$-degree of $f_T$, let $n$ be the $Y$-degree of $f_T$, and define $\Upsilon \subset \mathbb{C}^\times$ by

$$\Upsilon \;=\; \{\zeta \in \mathbb{C}^\times : \zeta^k = 1 \text{ for some } 1 \le k \le 2mn\}.$$

**Lemma 3.3** *$V(I) \subset (\mathbb{C}^\times \times \Upsilon) \cup (\Upsilon \times \mathbb{C}^\times)$.*

*Proof:* Let $(\alpha,\beta) \in V(I)$, and suppose neither $\alpha$ nor $\beta$ is in $\Upsilon$. By Lemma 2.7 and Lemma 3.2 we have

$$0 \;=\; f_{T(k)}(\alpha,\beta) \;=\; f_T(\alpha^k,\beta^k) \;=\; (\alpha^k-1)(\beta^k-1)f_T^*(\alpha^k,\beta^k)$$

for all $k \ge 1$. Since $\alpha$ and $\beta$ aren't in $\Upsilon$ this implies $f_T^*(\alpha^k,\beta^k) = 0$ for $1 \le k \le 2mn$. Therefore by Lemma 3.4 below there exist $c, d \in \mathbb{Z}$ such that $f_T^*(X^c, X^d) = 0$. It follows that $f_T^*(1,1) = 0$, contrary to Lemma 3.2. We conclude that if $(\alpha,\beta) \in V(I)$ then at least one of $\alpha, \beta$ must be in $\Upsilon$. $\square$

**Lemma 3.4** *Let $K$ be a field and let $f^* \in K[X,Y]$ be a nonzero polynomial with $X$-degree $m-1$ and $Y$-degree $n-1$. Assume there are $\alpha, \beta \in K^\times$ such that*

1. *$\alpha$ and $\beta$ are not $k$th roots of 1 for any $1 \le k \le 2mn$, and*
2. *$f^*(\alpha^k, \beta^k) = 0$ for all $1 \le k \le 2mn$.*

*Then there exist relatively prime integers $c, d$ with $1 \le c \le n-1$ and $1 \le |d| \le m-1$ such that $f^*(X^c, X^d) = 0$.*

*Proof:* Define an $mn \times mn$ matrix $M$ whose columns are indexed by pairs $(i,j)$ with $0 \le i \le m-1$ and $0 \le j \le n-1$ by letting the $k$th entry in the $(i,j)$ column of $M$ be $\alpha^{ik}\beta^{jk}$. Since $f^*(\alpha^k, \beta^k) = 0$ for $1 \le k \le mn$, the coefficients of $f^*$ give a nontrivial element of the nullspace of $M$. Since $M$ is essentially a Vandermonde matrix this implies

$$0 \;=\; \det(M) \;=\; \alpha^{nm(m-1)/2}\beta^{mn(n-1)/2} \cdot \prod_{(i,j)<(i',j')} (\alpha^{i'}\beta^{j'} - \alpha^i\beta^j)$$

for an appropriate ordering of the pairs $(i,j)$. It follows that $\alpha^{i'}\beta^{j'} = \alpha^i\beta^j$ for some $(i',j') \neq (i,j)$, so $\alpha^{d_0} = \beta^{c_0}$ for some $(c_0,d_0) \neq (0,0)$ with $|c_0| \leq n-1$ and $|d_0| \leq m-1$. The first assumption implies that $c_0 \neq 0$ and $d_0 \neq 0$, so we may assume without loss of generality that $c_0 \geq 1$.

Let $e = \gcd(c_0,d_0)$ and set $c = c_0/e$ and $d = d_0/e$. Then since $(\alpha^e)^d = (\beta^e)^c$ with $\gcd(c,d) = 1$ there is a unique $\gamma \in K$ such that $\gamma^c = \alpha^e$ and $\gamma^d = \beta^e$. Let $q$ be an integer such that $1 \leq q \leq 2mn/e$. Then by the second assumption we have

$$0 \;=\; f^*(\alpha^{eq},\beta^{eq}) \;=\; f^*(\gamma^{cq},\gamma^{dq}),$$

and so $f^*(X^c,X^d) \in K[X,X^{-1}]$ has zeros at $X = \gamma^q$ for $1 \leq q \leq 2mn/e$. If these zeros are not distinct then for some $1 \leq r \leq 2mn/e$ we have $\gamma^r = 1$ and hence $1 = \gamma^{cr} = \alpha^{er}$, which violates the first assumption. Therefore $f^*(X^c,X^d)$ has at least $\lfloor 2mn/e \rfloor$ distinct zeros. On the other hand the degree of the rational function $f^*(X^c,X^d)$ is at most $(m-1)|c| + (n-1)|d|$, and since $|c| = |c_0/e| \leq (n-1)/e$ and $|d| = |d_0/e| \leq (m-1)/e$ we have

$$(m-1)|c| + (n-1)|d| \;\leq\; 2(m-1)(n-1)/e \;<\; \lfloor 2mn/e \rfloor.$$

Therefore $f^*(X^c,X^d) = 0$. $\square$

Let $l \geq 1$ and recall that $g_l(X,Y) = (X^l-1)(Y^l-1)$ is the polynomial associated to an $l \times l$ square with lower left corner $(0,0)$. The set $V(g_l) \subset \mathbb{C}^\times \times \mathbb{C}^\times$ of zeros of $g_l$ is the union of the lines $X = \zeta$ and $Y = \zeta$ as $\zeta$ ranges over the $l$th roots of 1. It follows from Lemma 3.3 that if we choose $l$ appropriately (say $l = (2mn)!$) then $V(g_l) \supset V(I)$. This need not imply that $g_l$ is in $I$, but by Hilbert's Nullstellensatz [5, VII, Th. 14] we do have $g_l^k \in I$ for some $k \geq 1$.

To show there exists $l$ such that $g_l \in I$ we use the theory of *primary decompositions* (see, e. g., chapters 4 and 7 of [1]). Let $A$ be a commutative ring with 1. We say that the ideal $Q \subset A$ is a *primary ideal* if whenever $xy \in Q$ with $x \notin Q$ there exists $a \geq 1$ such that $y^a \in Q$. By the Hilbert basis theorem, $\mathbb{C}[X^{\mathbb{Z}},Y^{\mathbb{Z}}]$ is a Noetherian ring [1, Cor. 7.7]. Therefore there are primary ideals $Q_1,\ldots,Q_r$ in $\mathbb{C}[X^{\mathbb{Z}},Y^{\mathbb{Z}}]$ such that $I = Q_1 \cap \ldots \cap Q_r$ [1, Th. 7.13]. The radical ideal

$$P_i \;=\; \sqrt{Q_i} \;=\; \{f \in \mathbb{C}[X^{\mathbb{Z}},Y^{\mathbb{Z}}] : f^r \in Q_i \text{ for some } r \geq 1\}$$

of the primary ideal $Q_i$ is automatically prime, and is called the prime *associated* to $Q_i$. We may also characterize $P_i$ as the smallest prime ideal containing $Q_i$.

Since $I = Q_1 \cap \ldots \cap Q_r$ we need to show that there exists $l \geq 1$ such that $g_l \in Q_i$ for all $1 \leq i \leq r$. Observe that if $l \,|\, l'$ then $g_l \,|\, g_{l'}$. Therefore it is enough to show that for each $i$ there is $l_i$ such that $g_{l_i} \in Q_i$, since in that case we have $g_l \in I$ with $l = \operatorname{lcm}\{l_1,\ldots,l_r\}$. To accomplish this we first restrict the possibilities for the prime ideals $P_i$.

Let $q = (2mn)!$. We observed above that $g_q^k \in I$ for some positive integer $k$. Since $P_i \supset Q_i \supset I$ this implies that $g_q^k \in P_i$. Therefore some irreducible factor of

$$g_q(X,Y)^k \;=\; \prod_{\zeta^q=1} (X-\zeta)^k(Y-\zeta)^k$$

lies in the prime ideal $P_i$. It follows that $X-\zeta \in P_i$ or $Y-\zeta \in P_i$ for some $\zeta \in \mathbb{C}^\times$ such that $\zeta^q = 1$.

Assume without loss of generality that $X-\zeta \in P_i$. Then $P_i$ contains the prime ideal $(X-\zeta)$ generated by the irreducible polynomial $X-\zeta$. If $P_i \neq (X-\zeta)$ let $h$ be an element of $P_i$ which is not in $(X-\zeta)$. By dividing $X-\zeta$ into $h(X,Y)$ we see that $h(\zeta,Y) \in P_i$. Since $P_i$ is prime and $\mathbb{C}$ is algebraically closed this implies that some linear factor $Y-\alpha$ of $h(\zeta,Y)$ is in $P_i$. Therefore $P_i$ contains the maximal ideal $(X-\zeta, Y-\alpha)$, so in fact $P_i = (X-\zeta, Y-\alpha)$. Moreover, we must have $\alpha \neq 0$ since $Y$ is a unit in $\mathbb{C}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. It follows that if $X-\zeta \in P_i$ then either $P_i = (X-\zeta)$ or $P_i = (X-\zeta, Y-\alpha)$ for some $\alpha \in \mathbb{C}^\times$.

We will make repeated use of the following elementary fact about primary ideals.

**Lemma 3.5** *Let $Q$ be a primary ideal and set $P = \sqrt{Q}$. If $gh \in Q$ with $h \notin P$ then $g \in Q$.*

*Proof:* Since $h \notin P$ we have $h^a \notin Q$ for all $a \geq 1$. Therefore by the definition of primary ideal we have $g \in Q$. $\square$

Assume now that $P_i = (X-\zeta)$ with $\zeta^q = 1$. Then $X^q - 1$ has a simple zero at $X = \zeta$. Therefore by Lemma 2.7 and Lemma 3.2 we have

$$\begin{aligned} f_{T(q)}(X,Y) &= f_T(X^q, Y^q) \\ &= (X^q-1)(Y^q-1) f_T^*(X^q, Y^q) \\ &= (X-\zeta) h(X,Y) \end{aligned}$$

for some $h \in \mathbb{C}[X,Y]$. Moreover we have $h(\zeta, Y) \neq 0$, since otherwise $0 = f_T^*(\zeta^q, Y^q) = f_T^*(1, Y^q)$, which would imply $f_T^*(1,1) = 0$, contrary to Lemma 3.2. Therefore $h \notin P_i = (X-\zeta)$. It follows by Lemma 3.5 that $X-\zeta \in Q_i$, and hence that $g_q \in Q_i$.

Now assume $P_i = (X-\zeta, Y-\alpha)$. If $\alpha$ is an $r$th root of 1 for some $r \geq 1$ then $X^{qr}-1$ has a simple zero at $X = \zeta$ and $Y^{qr}-1$ has a simple zero at $Y = \alpha$. As in the previous case this implies

$$\begin{aligned} f_{T(qr)}(X,Y) &= (X^{qr}-1)(Y^{qr}-1) f_T^*(X^{qr}, Y^{qr}) \\ &= (X-\zeta)(Y-\alpha) h(X,Y) \end{aligned}$$

for some $h \in \mathbb{C}[X,Y]$. Since $f_T^*(\zeta^{qr}, \alpha^{qr}) = f_T^*(1,1) \neq 0$, we have $h(\zeta, \alpha) \neq 0$, and hence $h \notin P_i$. Applying Lemma 3.5 we get $(X-\zeta)(Y-\alpha) \in Q_i$, and hence $g_{qr} \in Q_i$. If $\alpha$ is not a root of 1 we may choose $r \geq 1$ so that $f_T^*(\zeta^{qr}, \alpha^{qr}) = f_T^*(1, \alpha^{qr}) \neq 0$, since $f_T^*(1,1) \neq 0$ implies that $f_T^*(1, Y)$ has only finitely many zeros. Then $X^{qr}-1$ has a simple zero at $X = \zeta$ and $Y^{qr}-1$ is nonzero at $Y = \alpha$. By an argument similar to those used above we have $f_{T(qr)}(X,Y) = (X-\zeta)h(X,Y)$ for some $h \in \mathbb{C}[X,Y]$ such that $h(\zeta,\alpha) \neq 0$. This implies $h \notin P_i$, so by Lemma 3.5 we get $X-\zeta \in Q_i$, and hence $g_q \in Q_i$.

We've shown now that for each $1 \leq i \leq r$ there is $l_i \geq 1$ such that $g_{l_i} \in Q_i$. Therefore we have $g_l \in I$ with $l = \operatorname{lcm}\{l_1, \ldots, l_r\}$. It follows from Corollary 2.6 that $T(\mathbb{N})$ $\mathbb{C}$-tiles an $l \times l$ square. To prove that $T(\mathbb{N})$ $\mathbb{Q}$-tiles a square it is sufficient to prove that $g_l$ is in

the ideal $I_0$ in $\mathbb{Q}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ generated by $T(\mathbb{N})$. Equivalently, we need to show that $g_l$ is in the $\mathbb{Q}$-span of the set

$$\mathcal{E} \;=\; \{X^i Y^j f_{T(k)} : i, j, k \in \mathbb{Z},\, k \geq 1\}.$$

We have shown that $g_l$ is in the $\mathbb{C}$-span of $\mathcal{E}$. Since $g_l$ and the elements of $\mathcal{E}$ are all in $\mathbb{Q}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$, and

$$\mathbb{C}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}] \;\cong\; \mathbb{Q}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}] \otimes_{\mathbb{Q}} \mathbb{C},$$

it follows immediately that $g_l$ is in the $\mathbb{Q}$-span of $\mathcal{E}$. This completes the proof of Theorem 3.1. $\square$

**Corollary 3.6** *Let $T$ be a $\mathbb{Z}$-weighted tile made up of rectangles whose corners all have rational coordinates. Assume that the weighted area of $T$ is not zero. Then there exists a positive integer $w$ such that $T(\mathbb{N})$ $\mathbb{Z}$-tiles a square with weight $w$.*

*Proof:* By Theorem 3.1 we know that $T(\mathbb{N})$ $\mathbb{Q}$-tiles a square $R$, so there are rational numbers $a_1, \ldots, a_n$ and tiles $T_1, \ldots, T_n$, each a translation of some $T(k_i) \in T(\mathbb{N})$, such that $R = a_1T_1 + \ldots + a_nT_n$. Let $w \geq 1$ be a common denominator for $a_1, \ldots, a_n$. Then $wR = wa_1T_1 + \ldots + wa_nT_n$, and $wa_i \in \mathbb{Z}$ for $1 \leq i \leq n$. Therefore $T(\mathbb{N})$ $\mathbb{Z}$-tiles $wR$. $\square$

# 4 Tiling with integer weights

Let $T$ be a $\mathbb{Z}$-weighted lattice tile, and assume that the weighted area of $T$ is not zero. By Corollary 3.6 we know that $T$ $\mathbb{Z}$-shapetiles a square with weight $w$ for some positive integer $w$. We wish to find necessary and sufficient conditions for $T$ to $\mathbb{Z}$-shapetile a square with weight 1. To express these conditions we need a definition. Given $\mu \in \mathbb{Q} \cup \{\infty\}$ we say that two lattice squares $S_{ij}$ and $S_{i'j'}$ belong to the same *$\mu$-slope class* if the line joining their centers has slope $\mu$. The tile $T$ can be decomposed into a sum $T = C_1 + \cdots + C_k$ of lattice tiles such that for each $i$ the unit lattice squares which make up $C_i$ all belong to the same $\mu$-slope class.

**Proposition 4.1** *Let $T$ be a $\mathbb{Z}$-weighted lattice tile and let $n$ be a positive integer. Let $c$ and $d$ be relatively prime integers and set $\mu = -c/d$. Then the $\mu$-slope classes of $T$ all have weighted area divisible by $n$ if and only if $f_T$ is an element of the ideal $((X^d - Y^c)(X-1)(Y-1), n(X-1)(Y-1))$ in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$.*

*Proof:* The $\mu$-slope classes of $T$ all have weighted area divisible by $n$ if and only if we can write $T = T_1 + nT_2$, where $T_1$ and $T_2$ are $\mathbb{Z}$-weighted lattice tiles such that the $\mu$-slope classes of $T_1$ all have weighted area zero. Write the decomposition of $T_1$ into its $\mu$-slope classes as $T_1 = C_1 + \cdots + C_k$. Since $\mu = -c/d$ with $c$ and $d$ relatively prime, the lattice squares $S_{ij}$ and $S_{i'j'}$ are in the same $\mu$-slope class if and only if $S_{i'j'}$ is the translation

of $S_{ij}$ by $(dr, -cr)$ for some $r \in \mathbb{Z}$. Therefore if $C_t$ is the $\mu$-slope class of $T_1$ containing $S_{ij}$ we have

$$f_{C_t}(X,Y) \;=\; g(X^dY^{-c})X^iY^j(X-1)(Y-1)$$

for some $g \in \mathbb{Z}[X^{\mathbb{Z}}]$. Since the weighted area of $C_t$ is zero we see that $0 = f^*_{C_t}(1,1) = g(1)$, which implies $X-1 \,|\, g(X)$. It follows that $(X^dY^{-c}-1)(X-1)(Y-1)$ divides $f_{C_t}$ for $1 \le t \le k$, and hence also that $(X^dY^{-c}-1)(X-1)(Y-1)$ divides $f_{T_1}$. Conversely, if $(X^dY^{-c}-1)(X-1)(Y-1)$ divides $f_{T_1}$, it is easy to check that the $\mu$-slope classes of $T_1$ all have weighted area zero. It follows that the $\mu$-slope classes of $T$ all have area divisible by $n$ if and only if we can write

$$f_T(X,Y) \;=\; (X^dY^{-c}-1)(X-1)(Y-1)h_1(X,Y) + n(X-1)(Y-1)h_2(X,Y)$$

for some $h_1, h_2 \in \mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. Since $Y^c$ is a unit in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ this is equivalent to $f_T \in ((X^d - Y^c)(X-1)(Y-1), n(X-1)(Y-1))$. □

**Theorem 4.2** *Let $T$ be a $\mathbb{Z}$-weighted lattice tile. Then $T$ $\mathbb{Z}$-shapetiles a square if and only if the following two conditions hold:*

1. *The weighted area of $T$ is not zero.*
2. *For every $\mu \in \mathbb{Q}^\times$ the gcd of the weighted areas of the $\mu$-slope classes of $T$ is 1.*

*Proof:* Let $T$ be a tile which satisfies conditions 1 and 2. To show that $T$ $\mathbb{Z}$-shapetiles a square it is sufficient by Corollary 3.6 to show that $T(\mathbb{N}) \cup \{wR\}$ $\mathbb{Z}$-tiles a square, where $R$ is an $l \times l$ square and $l, w$ are positive integers. Let $S = S_{00}$ be the unit lattice square with lower left corner $(0,0)$. If $T(\mathbb{N}) \cup \{wS\}$ $\mathbb{Z}$-tiles an $a \times a$ square then by rescaling we see that $T(\mathbb{N}) \cup \{wR\}$ $\mathbb{Z}$-tiles an $la \times la$ square. Therefore it is sufficient to show that $T(\mathbb{N}) \cup \{wS\}$ $\mathbb{Z}$-tiles a square. Let $J$ be the ideal in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ generated by $\{f_{T(k)} : k \in \mathbb{N}\} \cup \{w(X-1)(Y-1)\}$. By Corollary 2.6 it is sufficient to show that $g_l \in J$ for some $l \ge 1$.

By the Hilbert basis theorem $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ is a Noetherian ring. Therefore the ideal $J$ has a primary decomposition $J = Q_1 \cap \ldots \cap Q_t$. We need to show that there exists $l \ge 1$ such that $g_l \in Q_i$ for all $i$. As in the proof of Theorem 3.1 it is enough to show that for each $i$ there is $l_i \ge 1$ such that $g_{l_i} \in Q_i$. Let $P_i = \sqrt{Q_i}$ be the prime associated to $Q_i$, and suppose $w \notin P_i$. Then since $w(X-1)(Y-1) \in Q_i$, by Lemma 3.5 we see that $(X-1)(Y-1) = g_1$ is in $Q_i$. If $w \in P_i$ then since $P_i$ is a prime ideal it follows that $P_i$ contains a prime integer $p$ which divides $w$, and hence that $P_i \cap \mathbb{Z} = p\mathbb{Z}$.

For $f \in \mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ let $\overline{f} \in \mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ be the reduction of $f$ modulo $p$, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field with $p$ elements. Let $\overline{P}_i$ be the ideal in $\mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ consisting of the reductions modulo $p$ of the elements of $P_i$. Since $p \in P_i$ the ideal $\overline{P}_i$ is prime. Let $\overline{J} \subset \mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ be the ideal consisting of the reductions modulo $p$ of the elements of $J$. Then $\overline{J}$ is generated by $\{\overline{f}_{T(k)} : k \ge 1\}$. Since $P_i \supset J$, we have $\overline{P}_i \supset \overline{J}$.

Let $K$ be an algebraic closure of $\mathbb{F}_p$ and let $V(\overline{J}) \subset K^\times \times K^\times$ be the set of common zeros of the elements of $\overline{J}$. Let $m$ be the $X$-degree of $\overline{f}_T$, let $n$ be the $Y$-degree of $\overline{f}_T$, and define $\overline{\Upsilon} \subset K^\times \times K^\times$ by

$$\overline{\Upsilon} \;=\; \{\zeta \in K^\times : \zeta^k = 1 \text{ for some } 1 \le k \le 2mn\}.$$

**Lemma 4.3** $V(\overline{J}) \;\subset\; (K^\times \times \overline{\Upsilon}) \cup (\overline{\Upsilon} \times K^\times)$.

*Proof:* Let $(\alpha, \beta) \in V(\overline{J})$ and suppose neither $\alpha$ nor $\beta$ is in $\overline{\Upsilon}$. Then for $1 \le k \le 2mn$ we have

$$0 \;=\; \overline{f}_{T(k)}(\alpha, \beta) \;=\; \overline{f}_T(\alpha^k, \beta^k) \;=\; (\alpha^k - 1)(\beta^k - 1)\overline{f}_T^*(\alpha^k, \beta^k).$$

Since $\alpha$ and $\beta$ aren't in $\overline{\Upsilon}$ this implies that $\overline{f}_T^*(\alpha^k, \beta^k) = 0$ for $1 \le k \le 2mn$. Therefore by Lemma 3.4 there are relatively prime integers $c, d$ with $c \ge 1$ and $d \ne 0$ such that $\overline{f}_T^*(X^c, X^d) = 0$. Let $\mathcal{A}$ be the quotient ring $\mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]/(X^d - Y^c)$, and let $x, y$ denote the images of $X, Y$ in $\mathcal{A}$. Then $x$ and $y$ are units in $\mathcal{A}$ satisfying $x^d = y^c$ with $\gcd(c, d) = 1$, so there is $z = x^a y^b$ in $\mathcal{A}^\times$ such that $x = z^c$ and $y = z^d$. Therefore the image of $\overline{f}_T^*$ in $\mathcal{A}$ is given by $\overline{f}_T^*(x, y) = \overline{f}_T^*(z^c, z^d)$, which equals zero since $\overline{f}_T^*(X^c, X^d) = 0$. It follows that $X^d - Y^c$ divides $\overline{f}_T^*$, and hence that $f_T^*$ is in the ideal $(X^d - Y^c, p)$ in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. Therefore $f_T = (X-1)(Y-1)f_T^*$ is in the ideal

$$((X^d - Y^c)(X-1)(Y-1), p(X-1)(Y-1))$$

in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. Proposition 4.1 now implies that every $\mu$-slope class of $T$ has area divisible by $p$. This violates condition 2 of the theorem, so we have a contradiction. $\square$

Set $q = (2mn)!$ and let $V(\overline{g}_q) \subset K^\times \times K^\times$ be the set of zeros of $\overline{g}_q$. Since $X^q - 1$ has zeros at all elements of $\overline{\Upsilon}$, we have $V(\overline{g}_q) \supset (K^\times \times \overline{\Upsilon}) \cup (\overline{\Upsilon} \times K^\times)$. Therefore Lemma 4.3 implies $V(\overline{g}_q) \supset V(\overline{J})$. Since $\overline{P}_i \supset \overline{J}$ we have $V(\overline{J}) \supset V(\overline{P}_i)$, and hence $V(\overline{g}_q) \supset V(\overline{P}_i)$. As in Section 3 Hilbert's Nullstellensatz implies that $\overline{g}_q^k \in \overline{P}_i$ for some $k \ge 1$. Since $\overline{P}_i$ is prime and

$$\overline{g}_q(X, Y)^k \;=\; (X^q - 1)^k (Y^q - 1)^k$$

we have either $X^q - 1 \in \overline{P}_i$ or $Y^q - 1 \in \overline{P}_i$. It follows that $P_i$ contains one of the ideals $(X^q - 1, p)$ or $(Y^q - 1, p)$. We may assume without loss of generality that $P_i \supset (X^q - 1, p)$.

By [1, Prop. 7.14] we have $Q_i \supset P_i^u$ for some $u \ge 1$. Therefore it is enough to prove that for every $u \ge 1$ there is $l \ge 1$ such that $g_l \in P_i^u$. Let $t$ be a positive integer. Expanding $X^{qt} - 1$ in powers of $X^q - 1$ gives

$$\begin{aligned} X^{qt} - 1 &= -1 + ((X^q - 1) + 1)^t \\ &= \sum_{j=1}^{t} \binom{t}{j} (X^q - 1)^j. \end{aligned}$$

If we choose $t$ to be divisible by a large power of $p$ then for small values of $j \geq 1$ the binomial coefficient $\binom{t}{j}$ is divisible by a large power of $p$. Thus every term in this expansion is divisible either by a large power of $p$ or a large power of $X^q - 1$. It follows that there exists $t \geq 1$ such that $X^{qt} - 1 \in (X^q - 1, p)^u$. Since $P_i^u \supset (X^q - 1, p)^u$ we get $g_{qt} \in P_i^u$, as required.

Assume conversely that $T$ $\mathbb{Z}$-shapetiles a square. Then the weighted area of $T$ is clearly not equal to zero, so condition 1 of Theorem 4.2 is satisfied. We need to show that for every $\mu \in \mathbb{Q}^\times$ the gcd of the weighted areas of the $\mu$-slope classes of $T$ is equal to 1. If we knew that the scale factors and the coordinates of the translation vectors used in shapetiling the square were all in $\mathbb{Z}$, or even in $\mathbb{Q}$, we could prove this using polynomials in $\mathbb{Z}[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$. Since we have no right to make this assumption, we need to work in the ring $\mathbb{Z}[X^{\mathbb{R}}, Y^{\mathbb{R}}]$.

We may assume that the square which is shapetiled by $T$ is $S = S_{00}$, the unit square with lower left corner $(0, 0)$. We have then $S = a_1T_1 + \cdots + a_kT_k$, where $a_i \in \mathbb{Z}$ and each $T_i$ is a translation of some $T(\rho_i)$. Let $p$ be prime and suppose that for some $\mu \in \mathbb{Q}^\times$ the areas of the $\mu$-slope classes of $T$ are all divisible by $p$. Let $c, d$ be integers such that $\gcd(c, d) = 1$ and $\mu = -c/d$. Let $\overline{f}_T \in \mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$ be the reduction of $f_T$ modulo $p$, and for $1 \leq i \leq n$ let $\overline{f}_{T_i} \in \mathbb{F}_p[X^{\mathbb{R}}, Y^{\mathbb{R}}]$ be the reduction of $f_{T_i}$. Then by Proposition 4.1 we see that $(X^d - Y^c)(X - 1)(Y - 1)$ divides $\overline{f}_T$ (in $\mathbb{F}_p[X^{\mathbb{Z}}, Y^{\mathbb{Z}}]$, and hence also in $\mathbb{F}_p[X^{\mathbb{R}}, Y^{\mathbb{R}}]$). Therefore by Lemma 2.7 and Lemma 2.4 we see that $\overline{f}_{T_i}$ is divisible by

$$(X^{\rho_i d} - Y^{\rho_i c})(X^{\rho_i} - 1)(Y^{\rho_i} - 1).$$

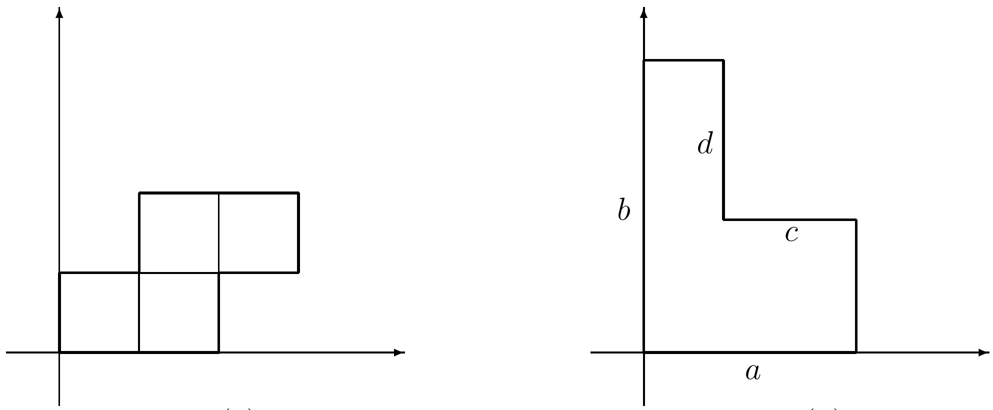
Define a ring homomorphism $\Psi : \mathbb{F}_p[X^{\mathbb{R}}, Y^{\mathbb{R}}] \to \mathbb{F}_p[X^{\mathbb{R}}]$ by setting $\Psi(f) = f(X^c, X^d)$. Since $\Psi(X^{\rho_i d} - Y^{\rho_i c}) = 0$, the divisibility relation from the preceding paragraph implies that $\Psi(\overline{f}_{T_i}) = 0$ for $1 \leq i \leq n$. On the other hand, since $\overline{f}_S = \overline{g}_1 = (X - 1)(Y - 1)$, we have

$$\Psi(\overline{f}_S) \;=\; X^{c+d} - X^c - X^d + 1,$$

which is nonzero since $c$ and $d$ are nonzero. Since $S = a_1T_1 + \cdots + a_kT_k$ we have $\overline{f}_S = \overline{a}_1\overline{f}_{T_1} + \cdots + \overline{a}_k\overline{f}_{T_k}$ with $\overline{a}_i \in \mathbb{F}_p$, which gives a contradiction. Therefore the areas of the $\mu$-slope classes of $T$ can't all be divisible by $p$, so condition 2 is satisfied. This completes the proof of Theorem 4.2. □

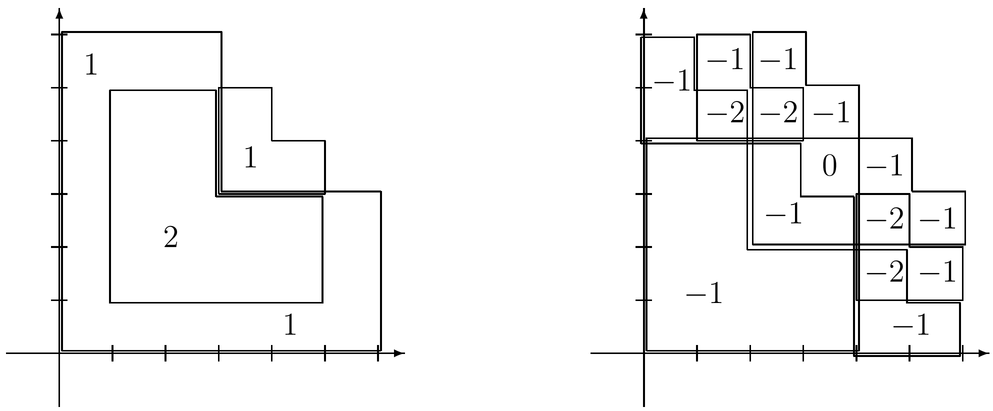
**Example 4.4** Let $T$ be the lattice tile pictured in Figure 2a. Since $T$ has area $4 \neq 0$, it follows from Theorem 3.1 that $T$ $\mathbb{Q}$-shapetiles a square. But since the nonempty 1-slope classes of $T$ both have area 2, Theorem 4.2 implies that $T$ does not $\mathbb{Z}$-shapetile a square.

**Example 4.5** Let $a, b, c, d$ be positive integers with $a > c$ and $b > d$. We construct a lattice tile $T$ by removing a $c \times d$ rectangle from the upper right corner of an $a \times b$ rectangle, as in Figure 2b. The area of $T$ is $ab - cd > 0$, so the first condition of Theorem 4.2 is satisfied. If $\mu > 0$ there is a $\mu$-slope class of $T$ consisting of just the




(a) (b)

Figure 2: Two tiles




Tiles with weight +1 Tiles with weight −1

Figure 3: A $\mathbb{Z}$-shapetiling of a square

upper left corner square, while if $\mu < 0$ there is a $\mu$-slope class of $T$ consisting of just the lower left corner square. In either case $T$ has a $\mu$-slope class whose area is 1. Therefore the second condition of Theorem 4.2 is also satisfied, so $T$ $\mathbb{Z}$-shapetiles a square.

**Example 4.6** The simplest case of Example 4.5 occurs when $a = b = 2$ and $c = d = 1$. In this case we have $f_T(X,Y) = (1+X+Y)(X-1)(Y-1)$. A straightforward calculation shows that

$$\begin{aligned} XY g_3(X,Y) \;=\; & (X^3Y^3 - X^2Y^2 - X^4 - X^4Y - X^4Y^2 - Y^4 - XY^4 - X^2Y^4) f_T(X,Y) \\ & + (XY-1) f_{T(2)}(X,Y) + f_{T(3)}(X,Y). \end{aligned}$$

This gives the $\mathbb{Z}$-tiling of a $3 \times 3$ square with lower left corner $(1,1)$ depicted in Figure 3. The left side of Figure 3 has tiles with weight 1 and the right side has tiles with weight $-1$. The total weights of the tiles covering each region are indicated.

# References

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Mass. (1969).

[2] F. W. Barnes, Algebraic theory of brick packing I, Discrete Math. **42** (1982), 7–26.

[3] M. Dehn, Über die Zerlegung von Rechtecken in Rechtecke, Math. Ann. **57** (1903), 314–332.

[4] K. Keating and J. L. King, Shape tiling, Electron. J. Combin. **4** (1997), no. 2, R12.

[5] O. Zariski and P. Samuel, *Commutative Algebra, Volume II*, Springer-Verlag, New York (1960).